Int. J. Solids. Structures, Vol. 18, No. 1, pp. 51–58, 1982
Printed in Great Britain.

0020-7683/82/010051-08$3.00/0

ELASTIC BUCKLING OF POLAR-ORTHOTROPIC ANNULAR PLATES IN SHEAR

DAVID DURBAN and YEHUDA STAVSKY
Department of Aeronautical Engineering, Technion-Israel Institute of Technology, Haifa, Israel

(*Received* 3 *March* 1981)

Abstract—An exact solution is presented for the elastic buckling of a polar-orthotropic annular plate subjected to uniform shearing stresses along the boundaries. Detailed results for the critical load are given for a wide range of material parameters, plate geometry and various boundary conditions. The limiting case of an infinite strip is also discussed and compared with earlier results for the buckling of infinitely long rectangular plates under shearing stresses.

INTRODUCTION

Elastic buckling of *isotropic* annular plates, under the action of uniform shearing stresses, has been investigated already by Dean[1] in 1924. A later paper by Federhofer and Egger[2] presents a study of the effect of radial thickness variations upon the critical load.

Here we show that Dean's exact solution can be extended to the case of *polar-orthotropic* plates with uniform thickness. The boundary conditions considered are any combination of clamped and simply supported edges. Detailed results for the buckling load are displayed over a wide range of material parameters and plate geometry.

Also, the *limiting problem* of an infinite strip is briefly discussed and its solution is shown to agree with earlier studies,[3–5], of *infinitely long rectangular* plates under shearing stresses.

ANALYSIS

An annular plate with inner radius a, outer radius b and uniform thickness h, is subjected to uniform self-equilibriated shearing stresses along the boundaries. The plate is elastic with polar-orthotropy described by the constitutive relations

$$\delta_2 = E_{rr}\epsilon_r + E_{r\theta}\epsilon_\theta \tag{1a}$$
$$\sigma_\theta = E_{r\theta}\epsilon_r + E_{\theta\theta}\epsilon_\theta \tag{1b}$$
$$\tau_{r\theta} = G\gamma_{r\theta} \tag{1c}$$

where, with the usual notation $(\sigma_r, \sigma_\theta, \tau_{r\theta})$ are the stress components in the polar system (r, θ), $(\epsilon_r, \epsilon_\theta, \gamma_{r\theta})$ are the associated engineering strain components and $(E_{rr}, E_{\theta\theta}, E_{r\theta}, G)$ the corresponding elastic moduli.† The origin of the polar system is located at the center of the plate.

It is now a matter of ease to verify that the prebuckling stress field is simply a state of *pure shear*, conveniently written as

$$\tau_{r\theta} = \frac{h^2}{24}\frac{\lambda}{r^2} \tag{2}$$

where λ is a constant. Note that the field (2) is *independent of material properties*.

Turning to the *buckling problem*, denoting by w the normal deflection during buckling, we

†Note that for an isotropic plate

$$E_{rr} = E_{\theta\theta} = \frac{E}{1-\mu^2},\quad E_{r\theta} = \frac{\mu E}{1-\mu^2},\quad G = \frac{E}{2(1+\mu)}$$

have by a standard formulation that

$$E_r\left(\frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}\right)-E_\theta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r^3}\frac{\partial w}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\left[2\left(\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w}{\partial r}+\frac{w}{r^2}\right)+E_\theta\left(\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2}+2\frac{w}{r^2}\right)\right]$$
$$-S\frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)=0 \quad (3)$$

where

$$(E_r, E_\theta, S)=\frac{(E_{rr}, E_{\theta\theta}, \lambda)}{E_{r\theta}+2G}. \quad (4)$$

If the material is isotropic then $E_r = E_\theta \equiv 1$ and (3) agrees with eqn (7) in[1].

The boundary conditions considered here are either a clamped edge where

$$w=0 \text{ and } \frac{\partial w}{\partial r}=0 \quad (5)$$

or a simply supported edge where

$$w=0 \text{ and } \frac{\partial^2 w}{\partial r^2}+\frac{\nu}{r}\frac{\partial w}{\partial r}=0 \text{ with } \nu=\frac{E_{r\theta}}{E_{rr}}. \quad (6)$$

Differential equation (3) in conjunction with boundary conditions (5), (6) form an *eigenvalue* problem for S.

The solution of (3) is now written in the form

$$w=Re\{r\phi(r)e^{im\theta}\} \quad m \text{ integer}. \quad (7)$$

Inserting (7) into (3) results in an ordinary differential equation for $\phi(r)$ with the solution

$$\phi(r)=A_1r^{x_1}+A_2r^{x_2}+A_3r^{x_3}+A_4r^{x_4} \quad (8)$$

where the a_i are integration constants and the x_i are the four *distinct* roots of the characteristic equation

$$E_rx^4-(E_r+E_\theta+2m^2)x^2+imSx+E_\theta(m^2-1)^2=0. \quad (9)$$

Compliance with boundary conditions—two at each edge—leads to four homogeneous algebraic equations with constants A_i as unknowns. A non-trivial solution (buckling) of that system is assured by equating to zero the 4×4 determinant formed by the coefficients of the system. The critical load (eigenvalue) is then obtained as the lowest value of S that will nullify the coefficient's determinant.

Denoting the radii ratio by

$$\rho=\frac{a}{b} \quad (10)$$

we find, after some algebraic manipulations, that the buckling equations for the boundary conditions considered here are as follows: (i) both edges clamped

$$\begin{vmatrix} \rho^{x_1} & \rho^{x_2} & \rho^{x_3} & \rho^{x_4} \\ x_1\rho^{x_1} & x_2\rho^{x_2} & x_3\rho^{x_3} & x_4\rho^{x_4} \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix}=0 \quad (11a)$$

(ii) inner edge clamped, outer edge simply supported

$$\begin{vmatrix} \rho^{x_1} & \rho^{x_2} & \rho^{x_3} & \rho^{x_4} \\ x_1\rho^{x_1} & x_2\rho^{x_2} & x_3\rho^{x_3} & x_4\rho^{x_4} \\ 1 & 1 & 1 & 1 \\ x_1(x_1+1+\nu) & x_2(x_2+1+\nu) & x_3(x_3+1+\nu) & x_4(x_4+1+\nu) \end{vmatrix}=0 \quad (11b)$$

(iii) outer edge clamped, inner edge simply supported

$$\begin{vmatrix} \rho^{x_1} & \rho^{x_2} & \rho^{x_3} & \rho^{x_4} \\ x_1(x_1+1+\nu)\rho^{x_1} & x_2(x_2+1+\nu)\rho^{x_2} & x_3(x_3+1+\nu)\rho^{x_3} & x_4(x_4+1+\nu)\rho^{x_4} \\ 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \end{vmatrix} = 0 \qquad (11c)$$

(iv) both edges simply supported

$$\begin{vmatrix} \rho^{x_1} & \rho^{x_2} & \rho^{x_3} & \rho^{x_4} \\ x_1(x_1+1+\nu)\rho^{x_1} & x_2(x_2+1+\nu)\rho^{x_2} & x_3(x_3+1+\nu)\rho^{x_3} & x_4(x_4+1+\nu)\rho^{x_4} \\ 1 & 1 & 1 & 1 \\ x_1(x_1+1+\nu) & x_2(x_2+1+\nu) & x_3(x_3+1+\nu) & x_4(x_4+1+\nu) \end{vmatrix} = 0. \qquad (11d)$$

NUMERICAL RESULTS

Equations (11) have been solved numerically over a wide range of representative material parameters (E_r, E_θ, ν) and plate geometry (ρ). The numerical scheme used is straightforward though somewhat laborious; for a given number of circumferential waves (m) an initial value of S is chosen and the corresponding four roots of eqn (9) are evaluated. The latter are used, with a given value of ρ to compute the value of the determinant (11). The procedure is then repeated iteratively with new values of S (but fixed m and ρ) until the smallest root of (11) is discovered. That root is finally minimized with respect to the integer m, yielding thus the critical values of S that will cause buckling of the plate. It is worth mentioning that in all cases treated here the roots of (9) were found to be distinct—in accord with representation (8). (The case of equal roots requires a separate treatment with the proper representation for $\phi(r)$).

Typical results for the critical value of S are shown in Figs. 1–4. For an isotropic plate with clamped edges our results check with those obtained by Dean (Fig. 1).

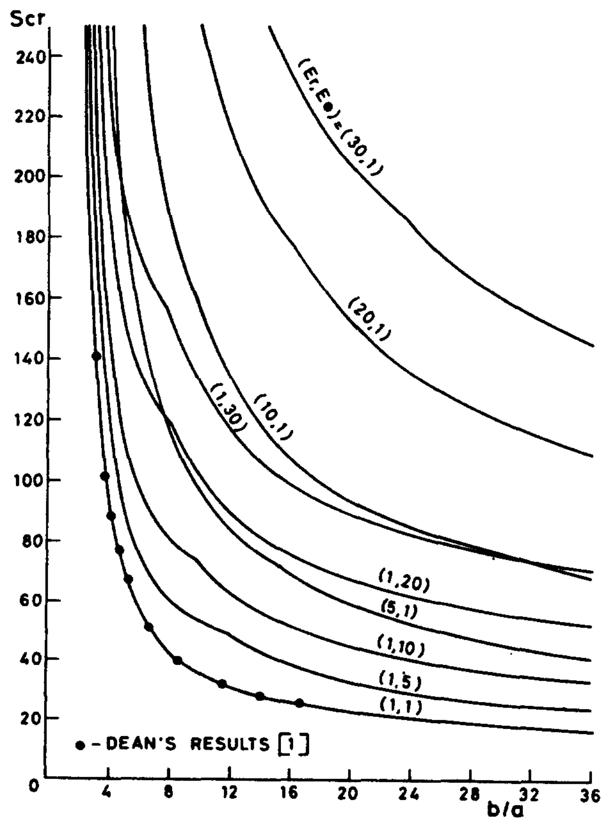

Fig. 1. Critical values of S. Both edges clamped.

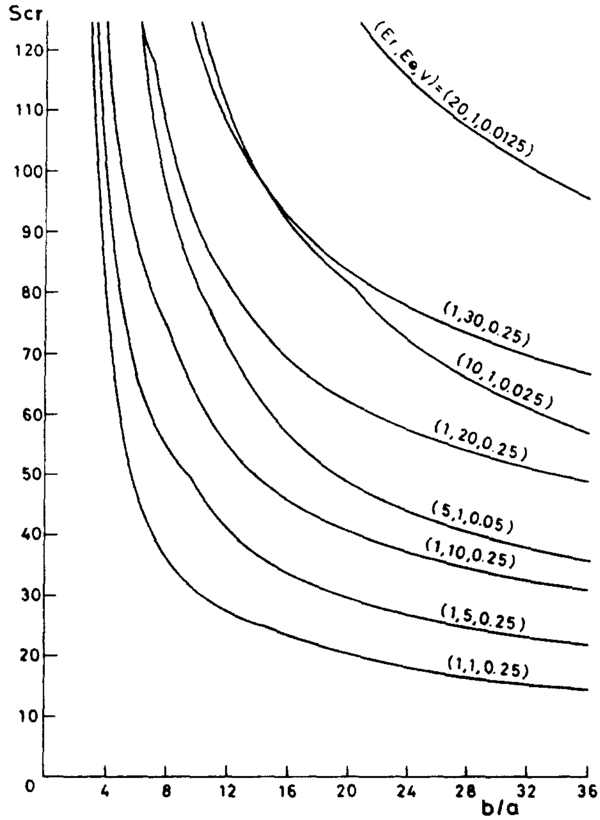

Fig. 2. Critical values of S. Inner edge clamped, outer edge simply supported.

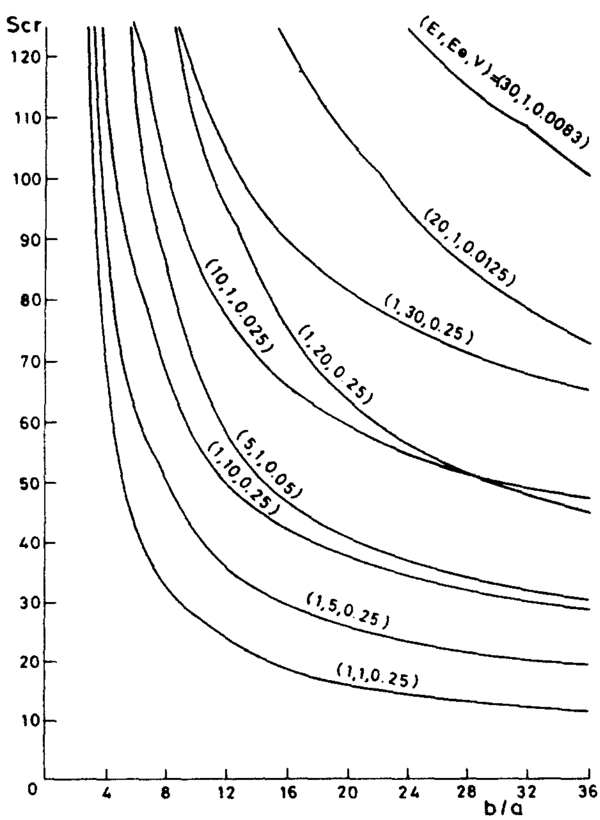

Fig. 3. Critical values of S. Inner edge simply supported, outer edge clamped.

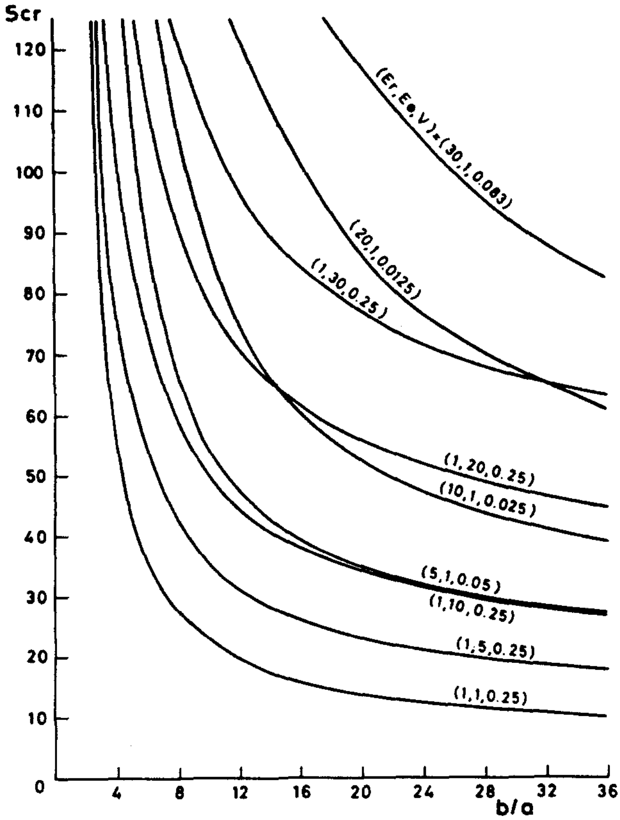

Fig. 4. Critical values of S. Both edges simply supported.

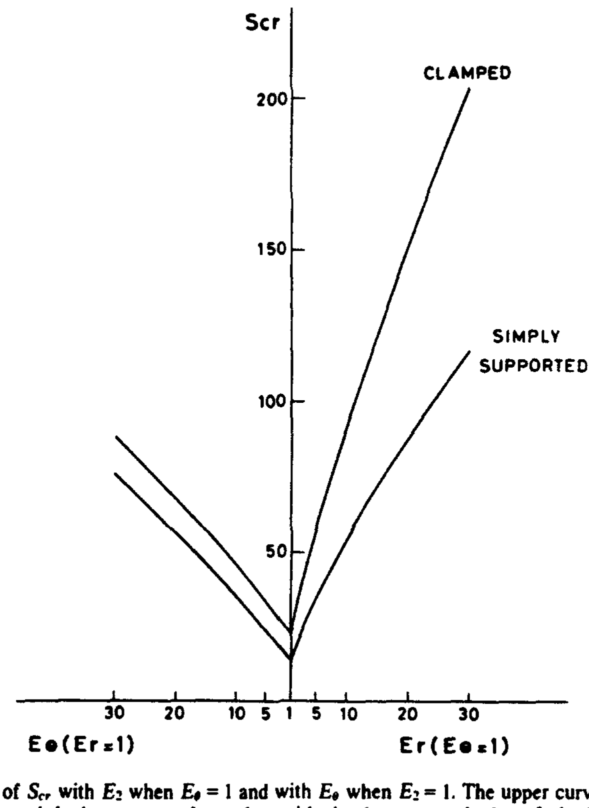

Fig. 5. Variation of S_{cr} with E_2 when $E_\theta = 1$ and with E_θ when $E_2 = 1$. The upper curve is for a plate with clamped edges and the lower curve for a plate with simply supported edges. In both cases $b/a = 20$.

Taking the isotropic case as a reference we see from Figs. 1–4 that, as expected, the buckling stress increases with any of the two parameters E_r, E_θ. *A more important conclusion is that increasing the radial parameter E_r has a greater effect on the critical load as compared to a similar increase of the circumferential parameter E_θ.* Just to give a few examples, for the clamped plate (Fig. 1) with $b/a = 5$ we have $S_{cr} \approx 192.5$ for $E_r = 5$ $E_\theta = 1$ and $S_{cr} \approx 90.5$ for $E_r = 1$ $E_\theta = 5$. Keeping the radial parameter constant ($E_r = 1$) we have to increase E_θ up to 30 (!) in order to obtain the same buckling stress as with $E_r = 5$ $E_\theta = 1$. Similarly, with $b/a = 31$ we have $S_{cr} \approx 74$ for $E_r = 10$ $E_\theta = 1$ but $S_{cr} \approx 35.5$ for $E_r = 1$ $E_\theta = 10$ – here again E_θ has to be increased up to 30 (with $E_r = 1$) in order to obtain buckling at the same stress level as with $E_r = 10$ $E_\theta = 1$. This effect is more emphasized in Fig. 5 which shows a typical dependence of S_{cr} on E_r with $E_\theta = 1$ and on E_θ with $E_r = 1$. These findings indicate the *advantage of using radial fibers* (or radial ribs) in strengthening composite annular plates against *shear-buckling*.

THE LIMITING PROBLEM

The buckling behaviour of the annular plate as $a/b \to 1$ should approach that of an infinitely long orthotropic rectangular plate subjected to shearing stresses at the boundaries. This limiting problem has been discussed by Dean[1] for isotropic plates and here we may essentially adopt his reasoning.

A direct evaluation of S_{cr} as $a/b \to 1$, using the previous procedure, becomes impossible since the wave number m, the roots of (9) x_i, as well as S_{cr} itself increase without limit. Instead, we introduce the geometrical parameter

$$\delta = \frac{b-a}{a} \tag{12}$$

so that $\delta \to 0$ as $a/b \to 1$. Now we define the quantities

$$\xi = x\delta \quad M = \frac{m\delta}{\sqrt{E_r}} \quad \sigma = \frac{S\delta^2}{\sqrt{E_r}} \tag{13}$$

expecting ξ, M, σ to remain finite as $\delta \to 0$.

Substituting definitions (13) into (9) and passing to the limit $\delta \to 0$ gives the characteristic equation

$$\xi^4 - 2M^2\xi^2 + iM\sigma\xi + \bar{E}^2M^4 = 0 \tag{14}$$

where

$$\bar{E} = \sqrt{E_r E_\theta}. \tag{15}$$

Likewise, it is a straightforward task to obtain the limiting forms of eqns (11). Observing the simple limit

$$\rho^x \to e^{-\xi} \text{ as } \delta \to 0 \tag{16}$$

we find that the limiting buckling equations are as follows:

(i) both edges clamped

$$\begin{vmatrix} e^{\xi_1} & e^{\xi_2} & e^{\xi_3} & e^{\xi_4} \\ \xi_1 e^{\xi_1} & \xi_2 e^{\xi_2} & \xi_3 e^{\xi_3} & \xi_4 e^{\xi_4} \\ 1 & 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{vmatrix} = 0 \tag{17a}$$

(ii) one edge clamped the other simply supported

$$\begin{vmatrix} e^{\xi_1} & e^{\xi_2} & e^{\xi_3} & e^{\xi_4} \\ \xi_1^2 e^{\xi_1} & \xi_2^2 e^{\xi_2} & \xi_3^2 e^{\xi_3} & \xi_4^2 e^{\xi_4} \\ 1 & 1 & 1 & 1 \\ \xi_1 & \xi_2 & \xi_3 & \xi_4 \end{vmatrix} = 0 \tag{17b}$$

(iii) both edges simply supported

$$\begin{vmatrix} e^{\xi_1} & e^{\xi_2} & e^{\xi_3} & e^{\xi_4} \\ \xi_1^2 e^{\xi_1} & \xi_2^2 e^{\xi_2} & \xi_3^2 e^{\xi_3} & \xi_4^2 e^{\xi_4} \\ 1 & 1 & 1 & 1 \\ \xi_1^2 & \xi_2^2 & \xi_3^2 & \xi_4^2 \end{vmatrix} = 0. \tag{17c}$$

The ξ_i are here the four distinct roots of the characteristic equation (14). Note that the moduli ratio $\nu = E_{r\theta}/E_{rr}$ does not appear now in the buckling equations.

The critical values of σ have been computed from (17) and (14) by the same numerical procedure as the one used for the annular plate. The main differences between the two problems are that σ_{cr} depends on a single material parameter $\bar{E}$ and that M is now a *continuous* variable.

To relate the critical eigenvalue σ_{cr} to the applied shear stress (2), we note that at the limit

$$r\delta \to l \text{ as } \delta \to 0 \tag{18}$$

where $l = b - a$ stands for the width of the plate. Combining now (2) with (4) and the third of (13), and observing (18), gives the relation

$$\tau_{r\theta} = \frac{1}{24}\left(\frac{h}{l}\right)^2 \sqrt{E_{rr}(E_{r\theta} + 2G)}\, \sigma. \tag{19}$$

Thus, once σ_{cr} is known the buckling stress is readily obtained from (19).

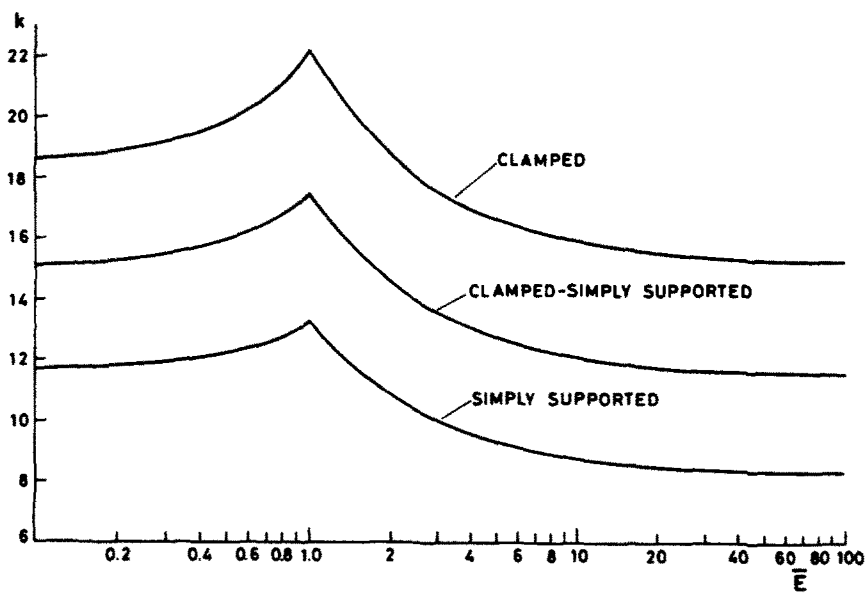

Fig. 6. Critical values of k. Infinitely long rectangular plate.

The numerical results are displayed in Fig. 6. For the sake of comparison with earlier work a new factor k is introduced so that

$$8k = \begin{cases} \sigma & \text{if } \bar{E} \leq 1 \\ \sigma/\sqrt{(\bar{E})} & \text{if } \bar{E} \geq 1. \end{cases} \tag{20}$$

The isotropic points ($\bar{E} = 1$) in Fig. 6 agree with those obtained in[3]. The curves for orthotropic clamped and simply supported plates agree with earlier results presented in[4, 5], see also[6, 7]. The curve for the clamped-simply supported plate appears to be new.

Acknowledgement—Thanks are due to the Technion–V.P.R. Fund—Semiel Ostrow Research Fund for its financial support.

REFERENCES

1. W. R. Dean, *Proc. R. Soc.* **A106**, 268 (1924).
2. K. Federhofer and H. Egger, *Ingenieur-Archiv* **XIV** 155 (1943).
3. R. V. Southwell and S. W. Skan, *Proc. R. Soc.* **A105**, 582 (1924).
4. S. Bergmann and H. Reissner, *Z. Flugtech. u. Motorluftsch.* **20**, 475 (1929).
5. E. Seydel, *Z. Flugtech. u. Motorluftsch.* **24**, 78 (1933).
6. S. P. Timoshenko and J. M. Gere, *Theory of Elastic Stability*. McGraw-Hill, New York (1961).
7. P. S. Bulson, *The Stability of Flat Plates*. Chatto and Windus, London, 1970.